Optimal Keywork Bids in Search-Based Advertising with Stochastic Ad Positions

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1 Introduction

- Only 2.6 million internet users in 1990, but nearly 1.6 billion by 2010

\[2.6(1 + i)^{20} = 1600 \iff i = 38\% \text{ growth p.a.}\]
• Banner ads (not our concern!) Click -> pay money

• Keyword ads, or, search based advertising (our concern!)

• Search engine advertising is new (Google 2001)

• We focus on Google
• Basics of SBA (Keyword ads)

Search-Based Advertising (SBA)

A keyword results in,
– (i) **Organic search results**, 

– (ii) **Sponsored results** 

- **Keyword:** “**Bike Tours Italy**”

- Google holds an **instantaneous auction** (GSP) among the advertisers bidding on that keyword:
  
  – bid + “ad quality”

- If you click on a sponsored link,
  
  – You are directed to the advertiser’s website, and
  
  – Google charges the advertiser for the click.

- **Downside:** Huge bills -> Budget
If budget exceeded, the ad is not displayed for the rest of the day,

- **Problem**: Choose the optimal bid price(s) subject to budget

## 2 Literature Review

- Rusmeivichientong and Williamson (2006): Select keywords

- Devanur and Hayes (2009): Sorting bids (Google’s problem)

- Özlük and Cholette (2007): Optimal bid (not stochastic)
• Our work: (i) Stochastic model, ad position is random, (ii) budget-related constraints

3 Preliminaries

• Our model in Figure 1.
Figure 1: A simple influence diagram for the model.
3.1 Distribution of the Ad Position $X$

- Bid price $b$ is our d.v. [¢/click].

- For a fixed $b$, the ad position is a r.v. $X \equiv X(b)$

- Ads on the first page attract attention

- First page as “unit” interval $[0, 1]$. Use beta for $X$

$$f_X(x; b) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad (0 < x < 1)$$  \hfill (1)

- Figure 2 for beta when $a = 20$.

- High values of $a$ correspond to high competition
Figure 2: Three dimensional graph of the beta density $f_X(x; b)$ when $a = 20$. Note that for small (large) values of $b$, the density is left- (right)-skewed.
3.2 The Expected Value $E[(1 - X)^m]$

- An interesting property of the beta

$$G(b) = E[(1 - X)^m] = \frac{\Gamma(a + b)\Gamma(b + m)}{\Gamma(b)\Gamma(a + b + m)}. \quad (2)$$

- When $m = 1$, $G(b) = b/(a + b)$ is increasing concave

4 Factors Affecting the Expected Profit $\mathcal{P}(b)$

4.1 The Expected Revenue $\mathcal{R}(b)$

- Important factors
4.1.1 IPT: # Impressions Per Time (const., [impr/time])

- How often the ad displayed per day (keyword popularity)

4.1.2 CTR: Click-Through-Rate (r.v. [click/impr])

- “Number” (rate) of clicks per impression

- \( Y \equiv CTR \)

- \( f_Y(y \mid x) = [p(x)]^y[1 - p(x)]^{y-1}, \ y = 0, 1, \) [Bernoulli with parameter \( p(x) \)]

- \( p(x) = (1 - x)^m \) for \( m \geq 1 \)
4.1.3 CPT: # Clicks Per Time (r.v. [click/time])

- Let $Y_j \equiv CTR(j)$: CTR for the $j$th impression

- $V = \sum_{j=1}^{IPT} Y_j$ : # of clicks per time (binomial)

- $U \equiv (V \mid X = x) = \left( \sum_{j=1}^{IPT} Y_j \mid X = x \right)$ is normal with mean $\mu_U(x) = IPT \cdot p(x)$

- Ad positions closer to 1 have higher variability CPTs

  - Variability of CPT is thru its c.o.v. $cv(x) = \ell x^n$ where $\ell > 1$ and $n \geq 1$

- Write $k \equiv IPT$

**Proposition 1** The expected number of clicks per time is $E(CPT) = E(V) = kG(b)$. 
4.1.4  **RPC: Revenue Per Click** (r.v., [¢/click])

- Mean $\mu_W$ (and variance $\sigma_W^2$)

- Revenue per time

  - product of (i) the number of clicks per time, and
  (ii) the revenue per click, $W$

  $$R \equiv \left( \sum_{j=1}^{IPT} Y_j \right) \cdot RPC = V \cdot W$$

**Proposition 2**  *The expected revenue is*

$$\mathcal{R}(b) = E[R(b)] = \mu_W kG(b).$$

4.2  **Expected Cost**

- **Cost per time** is a r.v. [¢/time] (Recall: $V$ is # clicks/time)

  $$C(b) = b \cdot (CPT) = b \cdot (IPT \cdot CTR) = b \cdot V.$$
Proposition 3  The expected cost
\[ C(b) = E[C(b)] = bkG(b). \]

4.3 Expected Profit

- Since expected profit \( P(b) = R(b) - C(b) \), we have
  \[ P(b) = kG(b)(\mu_W - b). \]

- Optimization problem with budget constraint
  \[
  \max_{b \geq 0} \quad P(b) = R(b) - C(b) \\
  \text{s.t.} \quad C(b) \leq B.
  \]

- Once \( b \) is known, can find \( \Pr\{C(b) \geq B\} \)

Proposition 4  The expected profit \( P(b) \) unimodal in \( b \).
(Figure 3.)
Figure 3: The expected profit function $\mathcal{P}(b)$ is unimodal in $b$ with the (unconstrained) global maximizer at $b^0$. 
Corollary 1 When budget is exhausted, we find $b^*$ and shadow price $\lambda$ from $\mathcal{P}'(b) = \lambda C'(b)$, and $C(b) = B$.

5 The Probability of Random Cost Exceeding the Budget

- The realized total cost may exceed the budget

- We maximize $\mathcal{P}(b)$ s.t. the constraint $h(b) \equiv \Pr[C(b) \geq B] \leq \theta$

5.1 Probabilistic Constraint

- Conditional cost $S = (bV \mid X)$ is normal with

$$f_S(s \mid x) = \frac{1}{\sigma_S \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left[ \frac{s - \mu_S(x)}{\sigma_S} \right]^2 \right\}.$$
5.2 Efficient Frontier Analysis

- Vary \( b \) over a range and generate \( (\mathcal{P}(b), h(b)) \) -> efficient frontier

- See Table 1 for a summary.

6 Bid Prices for a Single Keyword

- Now consider the four models
Table 1: Description of the four models considered in this paper.

<table>
<thead>
<tr>
<th>No Budget Constraints</th>
<th>Budget Constraint</th>
<th>Probabilistic Constraint</th>
<th>Trade-off Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max \mathcal{P}(b)$</td>
<td>$\max \mathcal{P}(b)$</td>
<td>$\max \mathcal{P}(b)$</td>
<td>$(\mathcal{P}(b), h(b))$</td>
</tr>
<tr>
<td>s.t. $C(b) \leq B$</td>
<td>s.t. $h(b) \leq \theta$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6.1 No Budget Constraint

- Special case with $m = 1$.

- For this problem $\mathcal{P}(b) = kb(\mu_W - b)/(a + b)$

- Since $\mathcal{P}(b)$ is strictly concave, its maximizer is
  $$b^0 = \sqrt{a^2 + a\mu_W} - a > 0.$$

- Note that for mean revenue per click $\mu_W$,
  $$\frac{\partial b^0}{\partial \mu_W} = \frac{a}{2\sqrt{a(a + \mu_W)}} > 0.$$
6.2 Budget Constraint

- When there is a budget constraint $C(b) \leq B$ we have, for $m = 1$

$$b^* = \begin{cases} 
  b^0 = \sqrt{a^2 + a\mu_W} - a, & \text{if } C(b^0) \leq B \\
  \tilde{b} = \frac{B + \sqrt{B^2 + 4kBa}}{2k}, & \text{if } C(b^0) > B
\end{cases}$$

6.3 Probabilistic Constraint

- Consider now $h(b)$ where

$$h(b) = \int_0^1 \left[ \frac{1}{2} - \frac{1}{2} \text{erf} \left( \frac{B - \mu_S(x)}{\sqrt{2}\sigma_S(x)} \right) \right] f_X(x, b) \, dx \leq \theta. \quad (4)$$

and, $\mu_S(x) = b\mu_U(x)$, and $\sigma_S^2 = b^2\sigma_U^2(x)$,

- When $m = 1$, let $\tilde{b} = h^{-1}(\theta)$, and

$$b^* = \begin{cases} 
  b^0 = \sqrt{a^2 + a\mu_W} - a, & \text{if } h(b^0) \leq \theta \\
  \tilde{b} = h^{-1}(\theta), & \text{if } h(b^0) > \theta
\end{cases}$$
6.4 Efficient Frontier Analysis and Trade-off Solutions

- Simply evaluate \((\mathcal{P}(b), h(b))\) for each feasible value of \(b\).

- Then, choose the "ideal" combination \((\mathcal{P}(b), h(b))\) and find \(b\).

6.5 Example with a Single Keyword

**Example 1** A single keyword problem with 
\[
\left[ a, k, \ell, m, n \mid \mu_W, B \mid \theta \right] = [20, 500, 5, 1, 1 \mid 50, 3000 \mid 0.10].
\]

With these data, the expected profit \(\mathcal{P}(b)\) is a concave function in Figure 4.
Figure 4: The expected profit function $\mathcal{P}(b) = E[P(b)]$ for the one keyword case in Example 1 reaches its unconstrained maximum at $b^0 = 17.4$ with $\mathcal{P}(b^0) = 7583.4$. Once the budget constraint $C(b) \leq B$ is taken into account, the constrained optimal solution is found as $\hat{b} = 14.3$ with $\mathcal{P}(\hat{b}) = 7447.3$. 
Figure 5: The probability \( h(b) = \Pr[C(b) \geq B] \) that the actual cost will exceed the budget \( B \) in Example 1.
Each point on this graph corresponds to a $(\mathcal{P}(b), h(b))$-pair in Example 1 for a fixed value of bid price $b$ which is varied from 1 to 50 in increments of 0.1.
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</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>17.4</td>
<td>14.3</td>
<td>4.97</td>
<td>7.0</td>
</tr>
<tr>
<td>( \mathcal{P}(b) )</td>
<td>7583.4</td>
<td>7447.3</td>
<td>4482.1</td>
<td>5574.1</td>
</tr>
<tr>
<td>( C(b) )</td>
<td>4053.5</td>
<td>3000</td>
<td>494.6</td>
<td>907.4</td>
</tr>
<tr>
<td>( h(b) )</td>
<td>0.53</td>
<td>0.49</td>
<td>0.10</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table 2: Numerical solution for the four problems with a single keyword and with parameter values \([a, k, \ell, m, n | \mu_W, B | \theta] = [20, 500, 5, 1, 1 | 50, 3000 | 0.10] \). For the constrained budget case (BC), the shadow price is found as \( \lambda = 0.28 \).

- Shadow price \( \lambda = 0.28 \) (for each cent increase in budget, the profit increases by approx. 0.28, or revenue by 1.28).
7 Analysis with Multiple Keywords

- In reality, many potential keywords to utilize (14,000?)

- When we consider multiple keywords,

\[
\mathcal{T}R(b) = \sum_{j=1}^{N} k_j \mu W_j G_j(b_j)
\]

\[
\mathcal{T}C(b) = \sum_{j=1}^{N} k_j b_j G_j(b_j)
\]

- If there is no budget constraint, and \( m = 1 \),

\[
\mathcal{T}P(b) = \sum_{j=1}^{N} \frac{k_j b_j (\mu W_j - b_j)}{a_j + b_j}
\]

\[
b_j^0 = \sqrt{a_j^2 + a_j \mu W_j} - a_j > 0, \quad j = 1, 2, \ldots, N.
\]
• If there is a budget constraint $\mathcal{T}C(b) \leq B$ and $\mathcal{T}C(b^0) > B$, then solve

$$\max_{b \geq 0} \mathcal{T}\mathcal{P}(b) \quad \text{s.t.} \quad \mathcal{T}C(b) = B \quad (6)$$

• Shadow price $\lambda$ for the budget constraint from the Lagrangian $\mathcal{L}(b, \lambda) = \mathcal{P}(b) - \lambda[C(b) - B]$

$$\nabla_b \mathcal{L}(b, \lambda) = 0, \quad \text{and} \quad \frac{\partial \mathcal{L}(b, \lambda)}{\partial \lambda} = 0, \quad (7)$$

• Probability of exceeding the budget

$$h(b) = \int_0^1 \cdots \int_0^1 \left[ \int_B^\infty f_S(s) \, ds \right] f_X(x, b) \, dx. \quad (8)$$

• If the solution of problem (6) results in an unacceptably high value for $h(b)$, then we would solve the new problem

$$\max_{b \geq 0} \mathcal{T}\mathcal{P}(b) \quad \text{s.t.} \quad h(b) = \Pr[\mathcal{T}C(b) \geq B] \leq \theta.$$
• A menu of \((P(b), h(b))\) values which can be used to pick a trade-off solution.

7.1 Example with Two Keywords

Example 2 Re-consider Example 1 with \(B = 3000\), but allow for a second keyword.

We use \([a, k, \ell, m, n \mid \mu_W, B \mid \theta] = [(20, 30), (500, 250), (5, 4), (1, 1), (1, 1) \mid (50, 30), 3000 \mid 0.10]\).

This KW #2 is less risky (lower CPT variance) but yields less return.
Surface of the total expected profit function $\mathcal{TP}(b)$ with two KWs in Example 2.
The probability $h(b) = \Pr[TC(b) > B]$ that the actual cost will exceed budget $B$ with two KWs in Example 2.
Each point on this graph corresponds to a $(TP(b), h(b))$-pair in Example 2 for a fixed value of bid prices $b = (b_1, b_2)$. 
Table 3: Numerical solution for the four problems with two keywords and parameter values \([a, k, \ell, m, n | \mu_W, B | \theta] = [(20, 30), (500, 250), (5, 4), (1, 1), (1, 1) | (50, 30), 3000 | 0.10]. For the constrained budget case (BC), the shadow price is found as \(\lambda = 0.48\).
• Sensitivity of the solution to $\theta$; see below

• Sharp decrease in expected profit for small $\theta$
The change in maximum expected profit

\[ \mathcal{T}\mathcal{P}(b^*) = E[TP(b^*)] \]

as \( \theta \) varies.
• Sensitivity of the difference $b_1^* - b_2^*$ to $\theta$; see below

• Very small $\theta$, bid more on the less risky keyword
The change in the difference in optimal bid prices $b_1^* - b_2^*$ as $\theta$ varies.
7.2 Three or More Keywords

- Problems 1 and 2 are easy for any $N$ (even thousands!)

- Problem 3: With $N \geq 3$, solution involving $h(b)$ is difficult:
  - A quadruple integral for $N = 4$: $20^4 = 160000$ computations in numerical integration.

8 Summary and Conclusions

- We maximized expected profit s.t. budget

- Shadow price for budget.
• Extensions:

  – Collecting actual data

  – Max $\Pr\{TP(b) \geq \text{Target Level}\}$ s.t. budget constraint

  – Efficient methods for Problem 3

  – Game- theory issues?